# Exact Finite Element Formulation of Frenet Formula of Curve in Geospatial Database 

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#### Abstract

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Directly using the difference form of Frenet formula will cause the three basic vectors losing their orthogonal features rapidly. As the analytic form is exact for infinite short length region, for finite length segregation, the omitted items should be retrieved to get high precision. Based on the unit orthogonal transformation in geometrical field theory, the Frenet formula is reformed for finite length region. Then, for given triple at the initial end of curve, using the curve parameters of curvature and torsion, the exact finite element formulation of Frenet formula is obtained to get the triples at any length position until the end point of curve. This method can be used as a high-precision technology for the measure, store, and retrieve of complicated curve.


## 1. Introduction

In navigation and trace (road) measurement and plotting in 3D space, how to exactly represent or retrieve a complicated curve with finite data sets is a theoretic and practical engineering problem ${ }^{[1-2]}$ in geospatial technology.
For an arbitral curve in 3D space, finite data are measured and stored. To display the whole curve or obtain exact features at some points in user defined coordinator system, data re-sampling is a standard procedure. How to get high precision is a practical problem. Generally speaking, directly using the Frenet formula to construct the calculation model is not practical. One apparent problem is that the analytical form is exact for infinite short length region, while the actual calculation has finite elements. Then, the size of the length will significantly produce error which may be accumulated to some points very rapidly. The well-known problem is that: directly use the difference form of Frenet formula ${ }^{[1-2]}$ will produce non-orthogonal triples. How to solve this problem is the topic for this paper.

To describe an arbitral curve in geospatial database, the usual way is to storage its coordinators at each length position $S$ as $(X(s), Y(s), Z(s))$. When the coordinator system is changed into a new coordinator system $(x(s), y(s), z(s))$, there is a relationship equation:

$$
\left|\begin{array}{c}
d x  \tag{1}\\
d y \\
d z
\end{array}\right|=\left|\begin{array}{ccc}
F_{X}^{x} & F_{Y}^{x} & F_{Z}^{x} \\
F_{X}^{y} & F_{Y}^{y} & F_{Z}^{y} \\
F_{X}^{z} & F_{Y}^{z} & F_{Z}^{z}
\end{array}\right|\left|\begin{array}{c}
d X \\
d Y \\
d Z
\end{array}\right|
$$

Where, the $F_{J}^{i}=\frac{\partial x^{i}}{\partial X^{J}}$ is the coefficients of local transformation (Here, $x^{1}=x, x^{2}=y, x^{3}=z ; \quad$ and $X^{1}=X, X^{2}=Y, X^{3}=Z$ ). Based on that, the curve coordinators data in new coordinators system is obtained

[^0]through calculation.
Although it is easy to find out the relation for initial and end points of the curve exactly, the whole curve coordinators are obtained through calculation based on the above formula. Hence, the accuracy is controlled by the calculation processes. Starting from one end point, for long curve or the high sampling curve, the error is additively transmitted to another end point. In geospatial technology, this problem is usually solved by redistribute the error among each points to sure the two ends coordinators are exact. Therefore, the accuracy is highly depends on data processing technology.
Is this way the unique way to describe an arbitral curve in 3D space? Definitely, the answer is no. If there is a new way can be acquired, what benefits can be achieved in geospatial database technology? To answer the last question, this paper will firstly introduce the mathematical formulation of arbitral curve in Frenet formula. Then, the tensor representation of curve is explained in mathematic formulation. Finally, the related measurements, storage, and display technology is discussed in theoretic sense. The tensor method to describe curve in geospatial database is proposed.

## 2. Basic Theory of Curve in 3D Space

For a curve in 3D space, the well-known Frenet formula ${ }^{[1]}$ is:

$$
\begin{align*}
& \frac{\partial \boldsymbol{\alpha}}{\partial s}=\frac{1}{\rho} \boldsymbol{\beta} \\
& \frac{\partial \boldsymbol{\beta}}{\partial s}=\frac{1}{\tau} \boldsymbol{\gamma}-\frac{1}{\rho} \boldsymbol{\alpha}  \tag{2}\\
& \frac{\partial \boldsymbol{\gamma}}{\partial s}=-\frac{1}{\tau} \boldsymbol{\beta}
\end{align*}
$$

Where, $s$ is the length parameter, $\boldsymbol{\alpha}$ is the tangent unit vector, $\boldsymbol{\beta}$ is the main normal unit vector, $\boldsymbol{\gamma}$ is the additional normal unit vector. For simplicity, here and after, letting $\mathbf{g}_{1}=\boldsymbol{\alpha}$, $\mathbf{g}_{2}=\boldsymbol{\beta}, \mathbf{g}_{3}=\boldsymbol{\gamma}$, the direct-difference form of Frenet formula is written as :

$$
\begin{gather*}
\mathbf{g}_{1}(s+d s)-\mathbf{g}_{1}(s)=\frac{1}{\rho} \mathbf{g}_{2}(s) \cdot d s \\
\mathbf{g}_{2}(s+d s)-\mathbf{g}_{2}(s)=\left[-\frac{1}{\rho} \mathbf{g}_{1}(s)+\frac{1}{\tau} \mathbf{g}_{3}(s)\right] \cdot d s  \tag{3}\\
\mathbf{g}_{3}(s+d s)-\mathbf{g}_{3}(s)=-\frac{1}{\tau} \mathbf{g}_{2}(s) \cdot d s
\end{gather*}
$$

For unit length increment $d s=1$, to get the tensor form, it is rearranged as:

$$
\left.\left|\begin{array}{l}
\mathbf{g}_{1}^{\text {end }}  \tag{4}\\
\mathbf{g}_{2}^{\text {end }} \\
\mathbf{g}_{3}^{\text {end }}
\end{array}\right|=\left|\begin{array}{ccc}
1 & \frac{1}{\rho} & 0 \\
-\frac{1}{\rho} & 1 & \frac{1}{\tau}| | \begin{array}{c}
\mathbf{g}_{1}^{\text {ini }} \\
\mathbf{g}_{2}^{\text {ini }} \\
0
\end{array} \\
-\frac{1}{\tau} & 1
\end{array}\right| \begin{aligned}
& \mathbf{g}_{3}^{\text {ini }}
\end{aligned} \right\rvert\,
$$

Where, the up-index 'ini' is used to refer the initial point $s$, the 'end' is used to refer the end point $s+d s$. Surely, for a long curve, the curvature $\frac{1}{\rho}$ and torsion $\frac{1}{\tau}$ are the functions of length. It is easy to find out that: the initial orthogonal unit features of $\mathbf{g}_{1}^{\text {ini }}, \mathbf{g}_{2}^{\text {ini }}, \mathbf{g}_{3}^{\text {ini }}$ cannot be conserved as the end point, as $\mathbf{g}_{1}^{\text {end }} \cdot \mathbf{g}_{3}^{\text {end }}=-\frac{1}{\rho \tau}$. Hence, it must be reformulated based on a more exact mathematic theory. In tensor theory, such a kind of unit orthogonal tensor ${ }^{[3]}$ is expressed as:

Where, the $\Theta$ is the local rotation angle of the incremental curve. Comparing with Eq. (4), it is easy to find out that:

$$
\begin{equation*}
L_{3} \sin \Theta=\frac{1}{\rho}, \quad L_{1} \sin \Theta=\frac{1}{\tau} \tag{6}
\end{equation*}
$$

Hence, the following results are obtained:

$$
\begin{aligned}
& \sin \Theta=\sqrt{\left(\frac{1}{\rho}\right)^{2}+\left(\frac{1}{\tau}\right)^{2}} \\
& L_{1}=\frac{1}{\sqrt{1+\left(\frac{\tau}{\rho}\right)^{2}}} \\
& L_{3}=\frac{1}{\sqrt{1+\left(\frac{\rho}{\tau}\right)^{2}}}
\end{aligned}
$$

Therefore, this research concludes that: 1) The Frenet formula is a first order approximation for calculating the triple vectors along the curve; 2) For finite length increment, the exact formulation is Eq.(5) which can be directly calculated without the problem of losing orthogonal feature of unit triple.

## 3. Operations in Geospatial Database

For an arbitral curve, taking its length as the 1D manifold intrinsic coordinator $S$, then the curve is completely determined by the curvature and torsion function $\rho(s), \tau(s)$. To plot the curve in 3D space, once the three basic vectors $\mathbf{g}_{1}\left(S_{0}\right), \mathbf{g}_{2}\left(S_{0}\right), \mathbf{g}_{3}\left(S_{0}\right)$ at a reference point $S_{0}$ are given in user's coordinator system, the whole curve can be retrieved using above finite element formulation. Without losing generality, if the stored curvature and torsion data are defined at length position $s_{k}, k=0,1,2, \cdots, N$, the user selection of starting point is defined by the three basic vectors $\mathbf{g}_{1}\left(S_{0}\right)=\mathbf{g}_{1}^{(0)}, \mathbf{g}_{2}\left(S_{0}\right)=\mathbf{g}_{2}^{(0)}, \mathbf{g}_{3}\left(S_{0}\right)=\mathbf{g}_{3}^{(0)}$, then the basic vectors at $S_{1}$ position is obtained by the following equation:

$$
\sin \Theta_{0}=\sqrt{\left(\frac{1}{\rho_{0}}\right)^{2}+\left(\frac{1}{\tau_{0}}\right)^{2}},
$$

$$
\begin{gather*}
L_{1}^{(0)}=\frac{1}{\sqrt{1+\left(\frac{\tau_{0}}{\rho_{0}}\right)^{2}}} \\
L_{3}^{(0)}=\frac{1}{\sqrt{1+\left(\frac{\rho_{0}}{\tau_{0}}\right)^{2}}} \\
\mathbf{g}_{1}^{(1)}=\left\{1+\left(1-\cos \Theta_{0}\right)\left[\left(L_{1}^{(0)}\right)^{2}-1\right]\right\} \mathbf{g}_{1}^{(0)}+ \\
L_{3}^{(0)} \sin \Theta_{0} \cdot \mathbf{g}_{2}^{(0)}+\left(1-\cos \Theta_{0}\right) L_{1}^{(0)} L_{3}^{(0)} \cdot \mathbf{g}_{3}^{(0)}  \tag{8}\\
\mathbf{g}_{2}^{(1)}=-L_{3}^{(0)} \sin \Theta_{0} \cdot \mathbf{g}_{1}^{(0)}+\cos \Theta_{0} \cdot \mathbf{g}_{2}^{(0)}+ \\
L_{1}^{(0)} \sin \Theta_{0} \cdot \mathbf{g}_{3}^{(0)} \\
\mathbf{g}_{3}^{(1)}=\left(1-\cos \Theta_{0}\right) L_{1}^{(0)} L_{3}^{(0)} \cdot \mathbf{g}_{1}^{(0)}-L_{1}^{(0)} \sin \Theta_{0} \cdot \mathbf{g}_{2}^{(0)}+ \\
\left\{1+\left(1-\cos \Theta_{0}\right)\left[\left(L_{3}^{(0)}\right)^{2}-1\right]\right\} \cdot \mathbf{g}_{3}^{(0)}
\end{gather*}
$$

Where, the index in brackets is for spatial positions. For position $s_{k}(N \geq k \geq 2)$, the general formulations are:

$$
\begin{aligned}
\sin \Theta_{k-1} & =\sqrt{\left(\frac{1}{\rho_{k-1}}\right)^{2}+\left(\frac{1}{\tau_{k-1}}\right)^{2}} \\
L_{1}^{(k-1)} & =\frac{1}{\sqrt{1+\left(\frac{\tau_{k-1}}{\rho_{k-1}}\right)^{2}}} \\
L_{3}^{(k-1)} & =\frac{1}{\sqrt{1+\left(\frac{\rho_{k-1}}{\tau_{k-1}}\right)^{2}}}
\end{aligned}
$$

$$
\begin{align*}
& \mathbf{g}_{1}^{(k)}=\left\{1+\left(1-\cos \Theta_{k-1}\right)\left[\left(L_{1}^{(k-1)}\right)^{2}-1\right]\right\} \mathbf{g}_{1}^{(k-1)}+ \\
& L_{3}^{(k-1)} \sin \Theta_{k-1} \cdot \mathbf{g}_{2}^{(k-1)}+\left(1-\cos \Theta_{k-1}\right) L_{1}^{(k-1)} L_{3}^{(k-1)} \cdot \mathbf{g}_{3}^{(k-1)} \tag{9}
\end{align*}
$$

$\mathbf{g}_{2}^{(k)}=-L_{3}^{(k-1)} \sin \Theta_{k-1} \cdot \mathbf{g}_{1}^{(k-1)}+\left[1-\left(1-\cos \Theta_{k-1}\right)\right] \mathbf{g}_{2}^{(k-1)}+$ $L_{1}^{(k-1)} \sin \Theta_{k-1} \cdot \mathbf{g}_{3}^{(k-1)}$
$\mathbf{g}_{3}^{(k)}=\left(1-\cos \Theta_{k-1}\right) L_{1}^{(k-1)} L_{3}^{(k-1)} \cdot \mathbf{g}_{1}^{(k-1)}-L_{1}^{(k-1)} \sin \Theta_{k-1} \cdot \mathbf{g}_{2}^{(k-1)}+$ $\left\{1+\left(1-\cos \Theta_{k-1}\right)\left[\left(L_{3}^{(k-1)}\right)^{2}-1\right]\right\} \cdot \mathbf{g}_{3}^{(k-1)}$

For higher precision, the re-sampling should be operation on raw data sets ( $\rho_{k}, \tau_{k}$ ). As the curvature and torsion are much more smooth functions than 3D coordinators or basic vectors, the digital filter technology or least-square technology should be applied there. In fact, the best way is to measure and store the curvature and torsion data for complicated curve rather than to store their coordinators data in measurement coordinator system. It is expecting that: measure, store, and retrieve an arbitral curve in 1D manifold intrinsic coordinator (length coordinator) will get high precision and high efficiency in data processing.

## 4. Why not Displacement Field

For geospatial data, usually the geodesy coordinator system is taken as the standard reference. So, the starting point basic vectors $\left(\mathbf{g}_{1}^{(0)}, \mathbf{g}_{2}^{(0)}, \mathbf{g}_{3}^{(0)}\right)$ are expressed by the local geodesy basic vectors $\left(\mathbf{e}_{r}\left(s_{0}\right), \mathbf{e}_{L}\left(s_{0}\right), \mathbf{e}_{B}\left(s_{0}\right)\right)$ in general form:

$$
\left|\begin{array}{l}
\mathbf{g}_{1}^{(0)}  \tag{10}\\
\mathbf{g}_{2}^{(0)} \\
\mathbf{g}_{3}^{(0)}
\end{array}\right|=\left|\begin{array}{lll}
a_{1 r} & a_{1 L} & a_{1 B} \\
a_{2 r} & a_{2 L} & a_{2 B} \\
a_{3 r} & a_{3 L} & a_{3 B}
\end{array}\right|\left|\begin{array}{l}
\mathbf{e}_{r}\left(s_{0}\right) \\
\mathbf{e}_{L}\left(s_{0}\right) \\
\mathbf{e}_{B}\left(s_{0}\right)
\end{array}\right|
$$

For a global curve, the local triple of curve $\mathbf{g}_{1}^{(k)}, \mathbf{g}_{2}^{(k)}, \mathbf{g}_{3}^{(k)}$ should be expressed by local geodesy basic vectors $\mathbf{e}_{r}\left(s_{k}\right), \mathbf{e}_{L}\left(s_{k}\right), \mathbf{e}_{B}\left(s_{k}\right)$. Therefore, after obtaining the curve expressed by $\mathbf{e}_{r}\left(S_{0}\right), \mathbf{e}_{L}\left(S_{0}\right), \mathbf{e}_{B}\left(S_{0}\right)$ (repeating use the finite element equations), they should be converted into the local geodesy basic vectors. This can be achieved by theoretic equation:

$$
\left|\begin{array}{l}
\mathbf{e}_{r}\left(s_{k}\right)  \tag{11}\\
\mathbf{e}_{L}\left(s_{k}\right) \\
\mathbf{e}_{B}\left(s_{k}\right)
\end{array}\right|=\left|\begin{array}{lll}
b_{1 r} & b_{1 L} & b_{1 B} \\
b_{2 r} & b_{2 L} & b_{2 B} \\
b_{3 r} & b_{3 L} & b_{3 B}
\end{array}\right|\left|\begin{array}{l}
\mathbf{e}_{r}\left(s_{0}\right) \\
\mathbf{e}_{L}\left(s_{0}\right) \\
\mathbf{e}_{B}\left(s_{0}\right)
\end{array}\right|
$$

Where, the tensors $\mathbf{a}$ and $\mathbf{b}$ are known or given quantities.

In geometrical field theory, it has proven that, for displacement field ( $u, v, w$ ) measured in local standard rectangular coordinator system ( $x, y, z$ ), the local rotation angle $\Theta$ is related with displacement gradient in form:

$$
\begin{equation*}
\sin \Theta=\frac{1}{2} \sqrt{\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)^{2}+\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)^{2}+\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)^{2}} \tag{12}
\end{equation*}
$$

Such a non-linear form denies the possibility to only use displacement field $(u, v, w)$ directly, except that their gradient is known quantities. In data management sense, for a point on curve, the displacement field (3 components) and their gradients ( 9 components) take up 12 data addresses, while the curvature and torsion take up 2 data addresses. This topic is too broad for this paper ${ }^{[4]}$, so our discussion will be limited as above.

## 5. Conclusion

The finite element formula of Frenet equations are obtained in this paper for an arbitral curve in 3D space. The research shows that: 1) the basic way to retrieve a complicated curve in 3D space is to tore its curvature and torsion as functions of curve length; 2) by a set of given initial triple at a length point, the triples on whole curve are determined by the finite element formulation in this paper; 3) the displacement field (or spatial position variation) is not suitable for the description of complicated curve as it requires too dense data for exact calculation of curve triples. The problem of how to exactly represent or retrieve a complicated curve with finite data sets as a theoretic and practical engineering problem in geospatial technology is answered.

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