Building a 4D Cell Complex Topology Through the Extrusion Process

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Keywords: Data Structures, Multidimensional Modelling, Spatial Models, Cell Complexes, Topology.

Abstract

This paper is a step forward towards a construction of 4D cell-complex models implemented using a topological data structure, dual half-edge. This allows for explicit representation of spatial relations among objects, such as adjacency relationship, coincidence or order of construction elements, e.g. vertices and edges. A method based on extrusion is proposed. An original 2D model is extruded to 3D, where adjacent cells sharing the same face are directly linked. Then extrusion to 4D is performed. In this case individual 4D cells sharing the same 3D cell are linked together. A model is kept valid at any stage of the construction process and links among cells are automatically provided by the construction operators. Implementation is based on the 4D data structure, which is used at each stage of construction, even in intermediate models of lower dimensionality.

1. Introduction

4D spatial models gain interest from various researchers in fields like 3D animation and tomography (Cavallo, 2021; Kwon and Zewail, 2010). They focus on spatio-temporal representation and analysis. Three spatial dimensions are accompanied by an additional one, which is time. Another closely related field is vario-scale modelling of objects, where, instead of time, scale is considered the fourth dimension. It is also possible to imagine a combined version integrating time and scale with a spatial model. Very often these extra dimensions are loosely connected with a well-defined spatial model using reference link, which may be difficult to maintain. Only a few representations propose strong topological connections between space and time/scale (Arroyo Ohori et al., 2015a; Caplan, 2024; van Oosterom et al., 2014).

Complexity of 4D models is still an issue in terms of computation and computer memory requirements. They have high requirements for storage resources. Extra dimensions bring exponential increase in the number of entities necessary to represent a model. For instance, a square, a basic 2D representation of a building, consists of four edges, while a 3D cube requires 12 edges, 4D hypercube – 32. Current technology development allows for implementation of such models, but spatial analysis methods and algorithms still need significant development. Even 3D computational geometry algorithms are still not perfectly robust and efficient as their 2D counterparts.

Efficiency of algorithms is also affected by implementation using a specific data structure. Those based only on the geometry are cheap in terms of memory storage but expensive in terms of spatial analysis, e.g. searching operations. On the other hand, topological data structures require much more storage while spatial analysis is much faster because of spatial relationships included explicitly in a model.

There are several data structures able to store the geometry and topology of 3D models, e.g. facet-edge (Dobkin and Laszlo, 1987), dual half-edge (DHE) (Boguslawski, 2011) or nD models, e.g. g-maps (Lienhardt, 1991). They allow for cell complex representation, where all cells are connected together using topological links. Generation of higher-dimensionality models is often realised in the extrusion process (Arroyo Ohori et al., 2015b), which introduces connections between entities in the highest dimension based on initial spatial relations in lower dimensions.

1.1 DHE data structure

DHE is a promising candidate for representation of 4D cell complexes (Boguslawski, 2024). The main advantage of the 3D version is implementation of the Poincaré duality, which reduces the number of construction elements to half-edges and nodes; introduction of an external cell enclosing the internal cells of the complex, which simplifies navigation operators; availability of operators for intuitive model construction and updates; and existence of simplified versions, which reduces storage requirements for models. The new 4D version should however introduce new construction operators, which are able implement the 4D duality concept (see Figure 1) in order to become a complete solution.

4D Poincaré duality applies to boundary representation models represented as graphs, and thus to models implemented using DHE. There are two graphs: primal and dual. They consist of nodes and edges. Edges divide space into regions, called faces. Each face has bounding edges, which form a cycle. 3D cells are bounded by faces, and 4D cells are bounded by 3D cells. In this paper, 3D cells are sometimes called 4D faces in order to emphasize their role as a bounding element. In the duality concept, each element has its corresponding element in the dual space. A primal node corresponds to the 4D cell in the dual graph (see Figure 1a), primal edge – to the dual 3D cell (see Figure 1b), primal face – to the dual face (see Figure 1c), primal 3D cell – to the dual edge (see Figure 1d), primal 4D cell – to the dual node (see Figure 1e).

Construction of spatial models, which have the boundary representation implemented with DHE, is performed using Euler operators in the CAD world (Boguslawski and Gold, 2016). These are specialised operators changing the geometry and topology in small steps. They introduce basic elements one by one. Different sequences of operators may be used to create

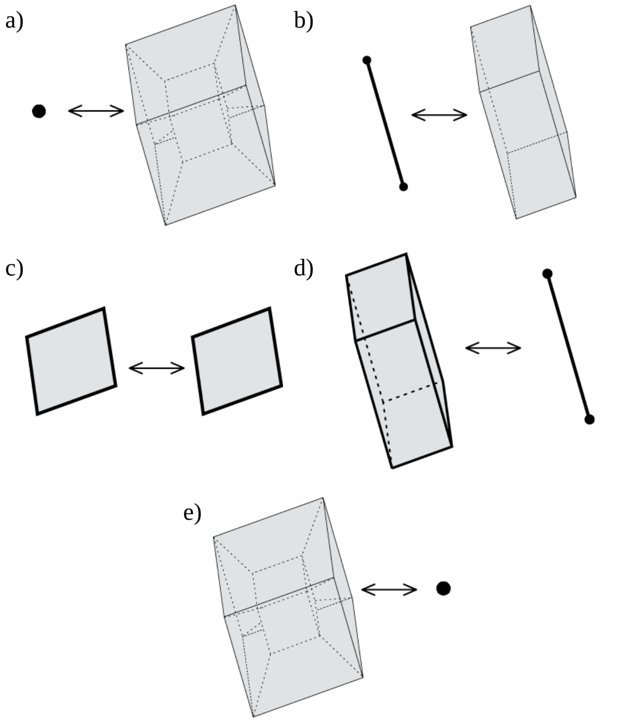

Figure 1 4D Poincare duality.

the same result model, for instance a polyhedron bounded by faces, which in turn are defined by bounding edges and vertices. Intermediate steps in the construction process can produce models, which may be difficult to imagine. This includes zero-length edges, non-planar faces, etc.

Another, more intuitive, method of model construction includes construction of faces, which are linked into polyhedra. It is similar to gluing cardboard pieces using sticky tape (Boguslawski, 2011). This cardboard and tape method produces the same result model as the method based on Euler operators but with different intermediate stages during construction.

The most important advantage of using construction operators is simultaneous maintenance of the geometry and topology. Using DHE brings also another challenge related to presence of the dual structure, which is responsible for the storage of spatial relations between cells in a complex. It increases the complexity of the model but decreases the number of entities necessary to represent it, namely edges and nodes.

There are several challenges that need to be faced in order to provide a complete method of a 4D model construction:

1. Implementation of the 4D Poincaré duality concept.
2. Simultaneous maintenance of the dual structure together with the changes of the geometry.
3. Introduction of the external cell.
4. Development of operators for incremental model construction.

Re 1) The Poincaré duality theorem is an essential base of the DHE data structure. Thanks to that there are only two atomic elements: half-edges and nodes, which are able to represent vertices, edges, faces, 3D and 4D cells. A 4D cell complex consists of many more atomic elements than a 3D complex. It is not enough to connect them using links describing the neighbourhood relation, but they must form certain topologies in the dual space.

Re 2) Introduction of new elements to the model requires updates of topological connections among half-edges. The dual structure should be updated at the same time with the changes of the geometry. The composition of connection in the final model can be determined relatively easily, but connections in the intermediate stages are difficult to imagine and visualise, e.g. due to the existence of non-manifold cases.

Re 3) An external cell encloses the internal cells and complements the model. It can be understood as 'the rest of the world' and prohibits 'falling from the edge' of the internal model. A 4D external cell, similarly to other individual 4D cells of the complex, is equivalent to a 3D cell complex. The issue related to the external cell is its visualisation, especially of the dual structure. The dual edges of the external cell are bounded by two nodes. One of them can have an attached location in 4D space, e.g. a point inside the 4D cell associated with the node. The second node however is located in the infinity, which is problematic for visualisation.

Re 4) 4D models require the introduction of more topological links than 3D models, because the number of entities, i.e. nodes, edges, faces and cells is higher. A single operator used in the construction process is updating more links and must assure their validity, which is especially difficult in case of dual links and the external cell mentioned in the items above.

This paper is a step forward towards an method for intuitive construction of 4D cell-complex models based on DHE. This includes geometry and topology updates which assure the model consistency at each step of construction. 4D cell complex construction based on introduction of individual 3D faces (i.e. cardboard and tape method) is proposed.

2. Methodology

In the first step all vertices of the 3D cell complex are generated and stored in a table. The remaining points of the 4D complex are computed later in the extrusion process. Starting from the anchor point P, there are M points added using a certain interval along the x axis. They form a one-dimensional vector of points. Then, all these points are copied N times and shifted by the interval along the y axis, which results in the $M \times N$ matrix. This process is repeated for the z axis, which results in $M \times N \times Q$ matrix of vertices.

The data structure used for implementation should be able to represent 4D cell complexes. In this research a new version of DHE derived from 3D DHE is used (Boguslawski, 2024). This data structure is applied for all models developed in this research, even in case of lower dimensionality models. The reason is that the result model is 4D and the others, i.e. 2D and 3D, are considered as intermediate steps.

The matrix is used to create the most bottom layer of 2D squares (see Figure 2a). They are then joined by shared edges. Square cells and links between them are created using specialised operators conforming to the cardboard and tape method. The cell complex is flat (no height), but due to the use of DHE, in terms of topology, it is a 3D model (see Figure 2b).

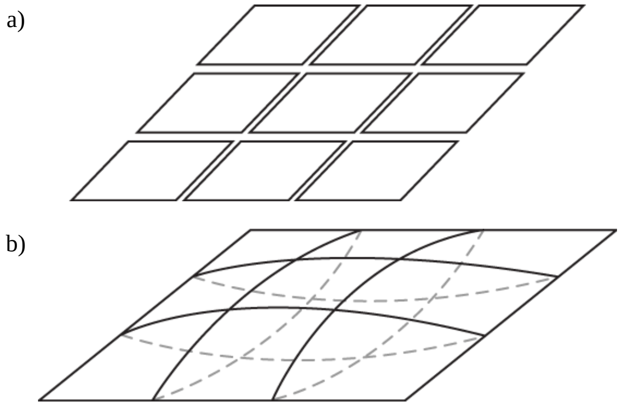

Figure 2 2D square cells: a) created based on a set of points; b) joined by shared edges into a cell complex.

In the next step, all vertical faces of the first layer are generated. After that, horizontal faces, i.e. top closing faces for individual cells, are created. It is shown in Figure 3a. Vertical faces as well as top faces of the bottom layer, which share edges bounded by the same vertices, are joined (see Figure 3b). It is important to notice that during joining a face with another face, shared edges from both faces are kept in the model. In result, two sides of a face are continuously separating with every join operation (see Figure 3c). When the horizontal face is added (see Figure 3d) and joined with the vertical faces, a 3D cell is formed (see Figure 3e). After all the faces are joined, a 3D cell complex is formed, where cells are connected together by 3D dual connections. The complex consists of internal cells, which are enclosed by one external cell (see Figure 3f). The above construction steps are repeated Q times for next layers of cells, until the $M \times N \times Q$ cell complex is created.

Until this moment, the construction process is almost the same as in the case of 3D models (Boguslawski, 2011). The most important difference is that the 4D dual structure is created and maintained by operators.

After the 3D cell complex is finished, it is extruded in the fourth dimension. In general, each face of a single cell is extruded to a 3D cell, which is considered a 4D face. In order to close the 4D cell, a 4D face is introduced on the top of vertical 4D faces. An example of a 4D cell, a tesseract, is shown in a schematic way in Figure 4a, while its projection on the 3D space is shown in Figure 4b. Faces of an original cube in the centre (bold lines) are marked with a number from 1 to 6. Each face is copied and used as a base for vertical 4D faces. New 3D faces are generated for each 3D edge of the original cube based on two original vertices and two extruded vertices, which were shifted by a certain interval into direction of the v axis – perpendicular to x, y and z axes. They are marked with numbers from 7 to 18. Then the closing 3D faces are added, which are also parts of an enclosing 4D face (grey lines). They are marked with numbers from 19 to 24. Eight vertices of this face have the same x, y and z coordinates as the original cube, but different v coordinate.

3D faces marked with the same numbers in Figure 4a are adjacent. They are generated once and after they are joined by edges with other faces, they are separated into individual 3D cells and thus forming faces of the 4D cell.

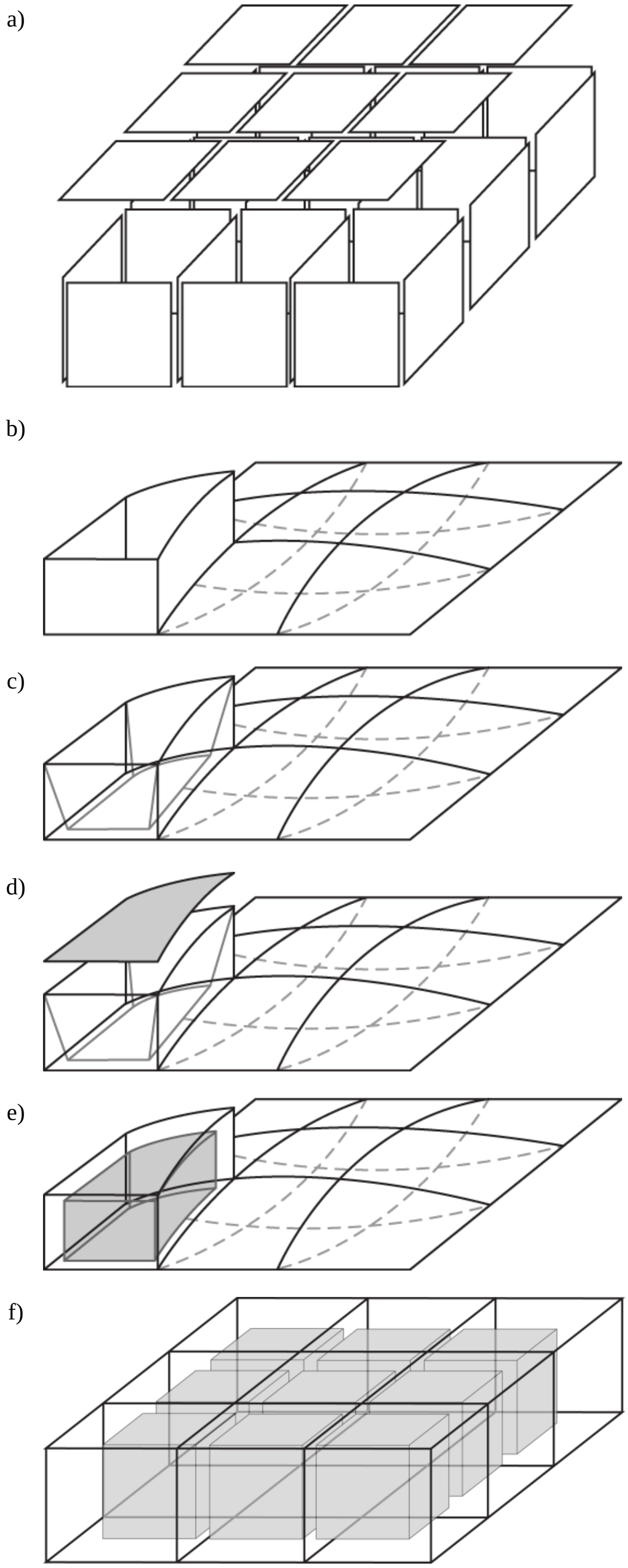

Figure 3 Extrusion process using the DHE data structure.

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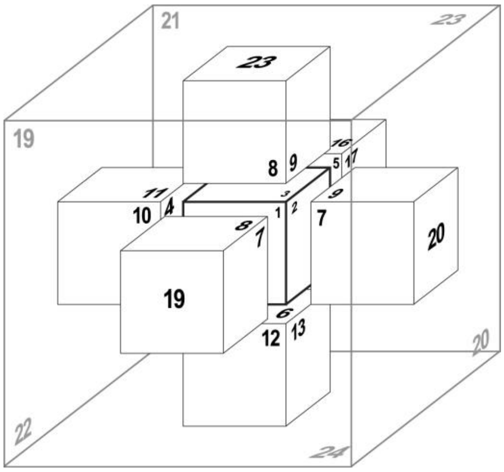
a)



b)

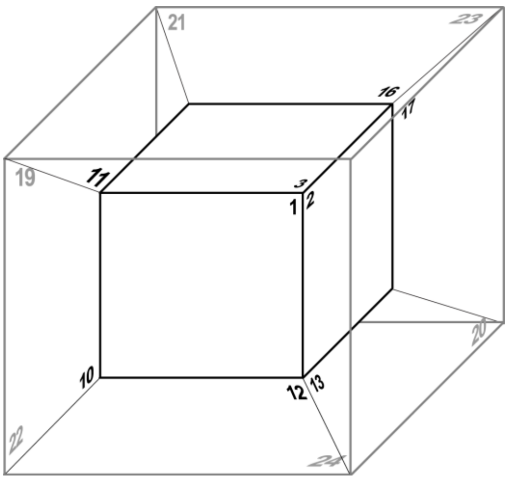


Figure 4 An example of a 4D cell, the tesseract: a) schematic representation; b) projection.

At the same time, when 4D faces are introduced in the extrusion process, links between adjacent faces are updated automatically in such a way that included edges are directly linked in the 4D dual. Links in the dual are based on the initial relationship between internal cells of the 3D cell complex (see grey cells in Figure 3f), but they are arranged to meet specific requirements of the Poincaré duality concept. 3D cells directly linked in the 3D dual space are disconnected after 4D extrusion. They are however connected via the new extruded 4D face (grey cube in Figure 5). It should be noted that this face consists of two 3D cells, which have the same geometry, but each of them belong to two different 4D cells.

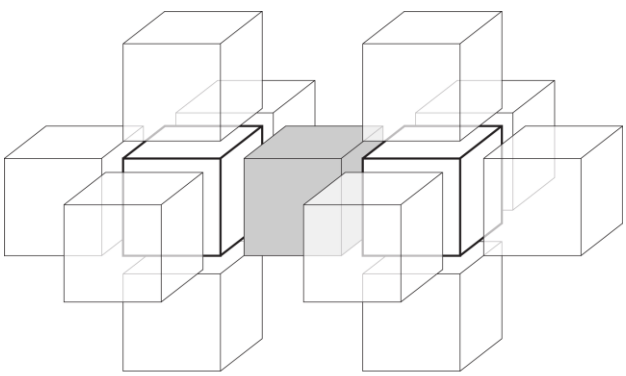


Figure 5 Two 4D cells joined by a shared 4D face (i.e. 3D cell).

In order to avoid deletion of any entities from the model during extrusion, a slightly different process is implemented for introduction of new vertical 4D faces between two cells, which were originally adjacent in the 3D cell complex, and between a cell and external cell.

In the case of two adjacent internal cells, i.e. grey cubes in Figure 3f, at the beginning of the 4D extrusion, there are two adjacent 3D faces created. They will belong to two adjacent cells, when a 3D cell is finally extruded. But before that they allow to keep a proper arrangement of links in the dual space. Two 4D faces are then independently built.

In the case of introduction of new vertical 4D faces between internal and external 3D cells, faces of the external cell are reused for construction of the 4D external cell.

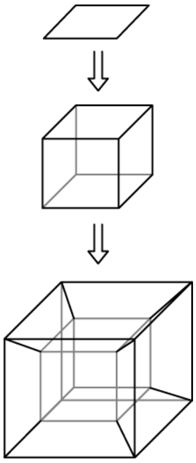
3. Results

Three models, cell complexes 1×1×1×1, 3×3×1×1 and 3×3×3×1 shown in Figure 6, were created. They consist of regular cells – cubes. The first one represents a single 3D cell extruded to hypercube/tesseract, the second and third ones consist of nine and 27 3D cells extruded in the fourth dimension respectively. However, the construction process is the same for all the models and is driven by the input parameters M, N and Q, which define the size of the cell complex and starting point P, which defines the anchor point for the whole model.

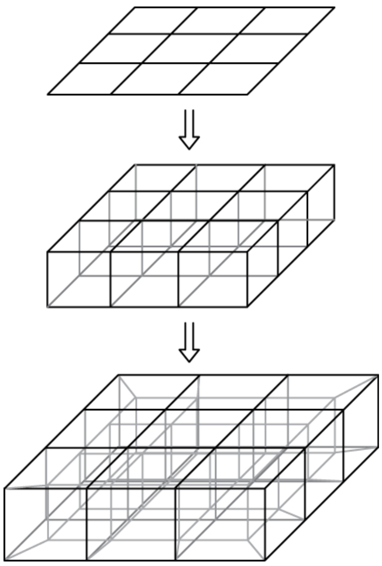
The number of entities in the result models are shown in Table 1. The first one, 1×1×1×1 consists of one internal and external 4D cell. Each of them consists of eight 3D cells. It is built using 32 edges with unique geometry, i.e. with unique bounding nodes. However, due to the use of DHE, each edge of the 4D cell is a bundle of three edges necessary for the proper representation of spatial relations between edges, but also for proper construction of the dual. Additionally, each DHE edge consists of two half-edges and each half-edge is associated with the dual half-edge. In order to compute the number of half-edges, the number of 3D cells in the complex (8) is multiplied by the number of edges in each 3D cell (12), multiplied by the number of half-edges in the edge (2), and because the DHE half-edge has the corresponding dual half-edge, it should be multiplied by 2. Finally, the external cell is taken into consideration. In case of a single 4D cell in the complex it is relatively easy – the number of elements should be multiplied by 2, as each 4D face has its corresponding external counterpart. In total, there are 768 half-edges used for this simple cell complex representing a tesseract. In this research a simplified version of the DHE was used, which reduced the number of pointers necessary for representation of the topology to five. It results in 3,840 pointers. In return, some limitations are introduced, which are not critical for this research: non-manifold cases are not allowed (only cell complexes, where cells are joined by a shared face are allowed), 3D dual is not present (adjacent 3D cells are directly joined by pointers). Only 4D dual is present in the model.

The second cell complex, 3×3×1×1 consists of nine internal cells. It is also enclosed by one external cell. But the number of external 4D faces is not proportional to the number of internal faces. Adjacent 4D cells are joined by 4D faces (i.e. 3D cells) of the same geometry. They are not part of the external cell. Thus, there are 48 external 3D cells. The ratio between the number of half-edges in this cell complex and the tesseract is 7.5, while there are nine times more internal cells.

a)



b)



c)

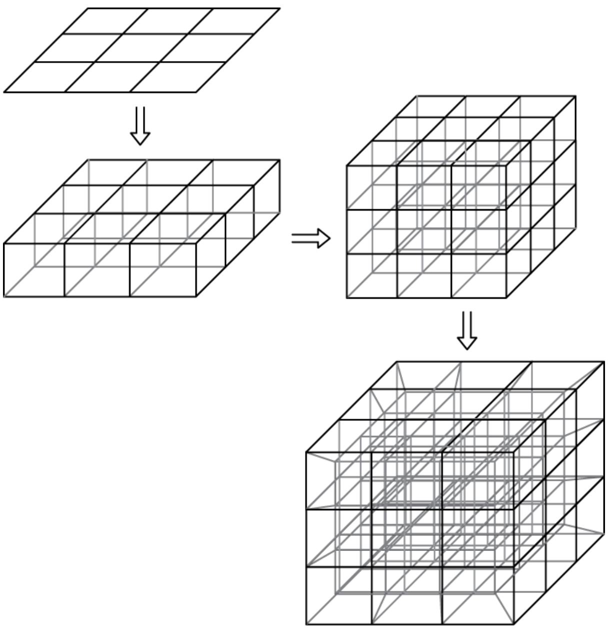


Figure 6 Cell complex construction process: a) 1×1×1×1 (tesseract); b) 3×3×1×1; c) 3×3×3×1.

The third cell complex, 3×3×3×1 consists of 27 internal cells, which are enclosed by one external cell consisting of 108 3D cells. In total, 15,552 half-edges are included, which is 20.25 times more than in the case of the first cell complex. This shows that the external cell consumes a relatively big amount of storage resources only in case of small cell complexes. With the bigger number of internal cells entirely surrounded by other internal cells, the storage cost of the external cells becomes smaller.

Entities	Cell complex		
	1×1×1×1	3×3×1×1	3×3×3×1
4D cells	1 + 1*	9 + 1*	27 +1*
3D cells	8 + 8*	72 + 48*	216 + 108*
edges	32	160	352
half-edges	768	5,760	15,552
pointers	3,840	28,800	77,760

Table 1 Number of entities in cell complexes. (* – cells belonging to the external cell)

4. Conclusions

The extrusion approach presented in this paper is an initial stage in our research, which is aimed at development of a construction method for 4D cell complexes with cells of an arbitrary shape.

Construction of a model and maintenance of the topology without dedicated operators is a tedious effort. Thus, operators for specific construction stages were developed in this research. The construction process is divided into a few steps. An initial 2D cell complex is created based on a set of vertices. The 2D model is then extruded in a way that each cell of a 2D complex is converted to a 3D cell linked with adjacent cells via shared faces. The next extrusion to 4D involves introduction of 3D cells which are shared between adjacent 4D cells.

Specialised operators are essential in a construction process. They provide an intuitive programming interface. They assure validity of a model and consistency between the geometry and topology. They were used to build different models: a simple one-cell complex (tesseract), and two cell complexes consisted of several cells. One of the applications that can be supported by these models is representation of building models in different levels of detail. The aforementioned cell complexes can be a virtual representation of a single room, a single floor in a building and the whole building respectively.

4D cell-complex models can be utilised in different applications, where spatio-temporal or vario-scale models are deployed. Time or scale can be represented as an additional spatial dimension, where 3D spatial layers are connected in the fourth dimension in the process of mapping of objects between the layers. This approach provides a mechanism for preserving consistency between consecutive layers representing time series or different scales. Any change introduced in a model at any location in time/scale can be easily reflected on the spatial models in the locations following each other.

This contribution has been peer-reviewed.
https://doi.org/10.5194/isprs-archives-XLVIII-4-2024-93-2024

Acknowledgements

This research work was funded in whole by the National Science Centre (NCN), Poland as part of the research program OPUS, project no.: 2021/41/B/ST10/03178.

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