FROM DATA TO TANGIBLE MODELS: CASE STUDY OF A VAULT IN THE ROYAL RESIDENCE OF VENARIA REALE

Fulvio Rinaudo[1], Vittorio Scolamiero[2], Ursula Zich[1], Martino Pavignano[1]

[1]Department of Architecture and Design, Politecnico di Torino – Italy
fulvio.rinaudo@polito.it; ursula.zich@polito.it; martino.pavignano@polito.it
[2]Interuniversity Department of Regional and Urban Studies and Planning, Politecnico di Torino – Italy
vittorio.scolamiero@polito.it

KEY WORDS: Royal Residence of Venaria Reale, LiDAR survey, Geometry, Dissemination, Physical model

ABSTRACT:

Framed on a wider research project that investigates Geometry as a cultural substrate and shared language for the comprehension of Architecture and its shapes, the presented research focuses on the geometric analysis and dissemination actions of a vault of the Royal Residence of Venaria Reale, designed by Benedetto Alfieri in the XVIII century. The vault is the only one offering to visitors' sight both its intrados and extrados surfaces. We propose an interdisciplinary approach that uses Geometry both as qualifying intangible heritage of the built shapes and as a language transversal to observation and survey, return of data and their interpretation from a dissemination point of view. To achieve this, we propose an innovative use of physical models, both in their meaning of object to be explored and in that of their design, between prototyping and seriality. Interaction between public and physical models becomes a way to promote critical shape-reading activities and to enhance spatial visualization abilities by their haptic/visual exploration, to recognize 3D built geometry and to explore architecture from different a point of view, getting closer to its shapes.

1. INTRODUCTION

Today,[1] instances related to new ways of communicating Cultural Heritage are intended as fundamental acts for their accessibility and valorisation. Hence, much research is looking at 'amplified usage' of every kind of heritage (Cetorelli and Guido, 2020). And these operations cannot be carried on «without a real awareness of the broader social layer that the Heritage itself is owner and responsible for» (Rudiero, 2019).
Within this context, we present a research project aimed at presenting Geometry as a cultural substrate and sharable language for the dissemination of Architectural Heritage, with the use of physical models as tools to analyse and communicate architectural artifacts. In fact, Architecture has always been inspired by Geometry and its concepts (Williams and Ostwald, 2017), thus it is important to highlight the important role of Geometry for the architectural shapes. Within the framework of our research, the choice of data to be communicated becomes crucial, so we investigate which of them should be used and how they could be represented.

2. METHODOLOGY

In this contribution we present the transition from 3D LiDAR survey data to a dissemination process of an architectural shape based on physical models. This is an opportunity to think about many topics between cultural matrices, sources, theoretical and built surfaces and communicative intentions. Moreover, the use of physical models allows to test hypotheses and to visualize them through a tangible approach that makes 'theoretical contents' more concrete: not only those related to the built but also the abstract ones, such as the theoretical shape. In this sense, we promote a critical and geometric re-reading of Cultural Heritage. Moreover L. March (2002) states that «architecture, in its applications, demands the concretization of abstract mathematical statements», hence we are looking for communicating such issues to a broader audience, within a dissemination perspective (Spallone and Vitali, 2017a), by promoting critical shape-reading activities to foster spatial prefiguration, in the sense of Leopold (2015). Moreover, the contribution recognises the values assumed by the physical models in the architectural field. In fact, despite the actual undisputed prevalence of the digital model, recently, perhaps also thanks to the diffusion of increasingly cheaper rapid prototyping systems, there has been a renewed interest in analogical architectural models (Frommel, 2014), especially in the field of Representation (Fatta, 2021; Cumino et al., 2022). This interest probably arises from the possibilities offered by digital tools and their interaction with systems for the creation of tangible artefacts starting from digital drawings. So, the relationship between the architecture to be represented and the model-form to be created in respect of the relationship between «the representative object-model and its variegated and multiple users» causes the possible «unity of techniques of planning and those of carrying out the planned works» for which a model is created (Ragazzo, 1997, p. 28). Thus, an approach to physical modelling that develops through the application of traditional and innovative prototyping techniques is considered of fundamental importance.

2.1 Physical models and cultural heritage accessibility

With the formalization of the principles of universal design thanks to the research of R. F. Mace in 1985, ideally preceded in the Sixties by the work of S. Goldsmith (1963), we witness

[1] While the research is the result of the collaboration between the authors, F. Rinaudo and V. Scolamiero wrote paragraph 4; V. Scolamiero wrote paragraphs 5; M. Pavignano wrote paragraphs 1, 2, 2.1, 6, 7.1, 7.2,; U Zich wrote paragraphs 3, 3.1, 3.2, 3.3, 7, 8.

This contribution has been peer-reviewed.
https://doi.org/10.5194/isprs-archives-XLVIII-M-2-2023-1315-2023

the gradual awareness in the world of culture and architectural design for the need to propose new ways of ‘using’ and ‘accessing’ everyday life. In the wake of these studies, we can place the Design for All (www.dfaeurope.eu), which at the beginning of the 2000s with the formulation of the Berlin Act (2005) and the Milan Charter (2007) helped to highlight the importance of Culture for All and Tourism for All. The underlying theme of these theories lies in the desire to make all aspects of human life freely accessible to all people, regardless of any ‘different abilities’. These abilities mainly refer to perceptive difficulties of various kinds, mainly attributable to the visual and motor fields. However, concepts of accessibility and inclusion have also opened their horizons to the sphere of perceptual difficulties related to the cognitive sphere, diversifying the approach between difficulties due to a lack of development of mental skills and a different level of development, referring in this case to the different phases of the cognitive development of the human being and, if desired, to the different intelligences as proposed by Gardner (1983).
Nowadays, the concept of perception as a set of *stimuli* of various kinds that bring man closer to the knowledge of ‘things’ is a much-investigated topic, especially from the point of view of visual perception or its deficits (Hachen, 2007).
The creation of tangible representations, also by means of rapid prototyping techniques, allows in this case to create artifacts that communicate directly with all the application areas of Design, proposing working methods aimed at the interaction between them, aimed at the creation of artifacts dedicated to a haptic perception of objects and places, see Sdegno et al. (2017), Empler at al. (2022). This attention is aimed above all at the definition of methods and strategies suitable for the design of tactile aids to support the processes of valorisation of cultural heritage, above all from the point of view of accessibility and inclusion, which are proving to be increasingly important issues in every field of human life (Azzolino and Lacirignola, 2019).

3. CULTURAL CONTEXT

We apply our research to a specific architectural fact in the museum context of the Royal Residence of Venaria Reale, which is one of the most important baroque complexes near Torino, Italy. Its construction dates from the second half of the XVII century onwards: it is the result of a stratification of different architectural layers, during at least one century and not only, for an exhaustive history of the building see Cornaglia (2006). The Palace was abandoned until 1999 when a great conservation/restoration/valorisation project started, under the financial help of the European Union and the Italian Government. Today, the Palace, which was recognized Unesco World Heritage in 1997 together with other Royal Sites in Torino and its suburbs, is home to many different museums and is a vital part of the cultural offer of the Piemonte Region (Cornaglia, 2018), being not only a place for cultural exhibitions but also an educational structure, according to the International Council of Museums Statutes, Vienna 2007.
In this contribution we focus on a vault designed by Benedetto Alfieri in the 40’s of the XVIII century, sited in one of the two spaces in the Residence (the other is Belvedere Tower, identified with the number 1 in Fig.1). This space was designed by Alfieri to connect the existing buildings, by rotating the main axis of the complex by 90 degrees (Bellini, 1979). The direct observation of the extrados of space marked number 2 in Fig. 1, and made visible after the restoration phase, allowed us to recognize some elements characterizing space marked with 2, that is we identified transit axes in the diagonals of the rhombuses which, as a theoretical figure, may be recognized as basis of the roofing systems design. These considerations supported us in survey and model design phases.
Such architectural element (Fig. 2), which might seem a simple ceiling, is the only vault inserted in permanent show offering to visitors not only its intrados surface (the ceiling of the connection space underneath) (Fig. 2c-d), but also its extrados structure (Fig. 2a-b), in the *foyer* of the *Sala dei Paggi*.
This condition, even if affected by an issue derived from the restoration project – the insertion of a double side ramp which covers ¼ of the structure – is quite an extraordinary opportunity to observe such and architectural element.
Still, this condition is not easily comprehensible, because the shapes of intrados and extrados apparently do not have a clear geometrical connection. Thus, it is interesting to investigate this vault with an interdisciplinary approach that makes use of Geometry as a language transversal to observation, survey and return of data, and their interpretation from a dissemination point of view.

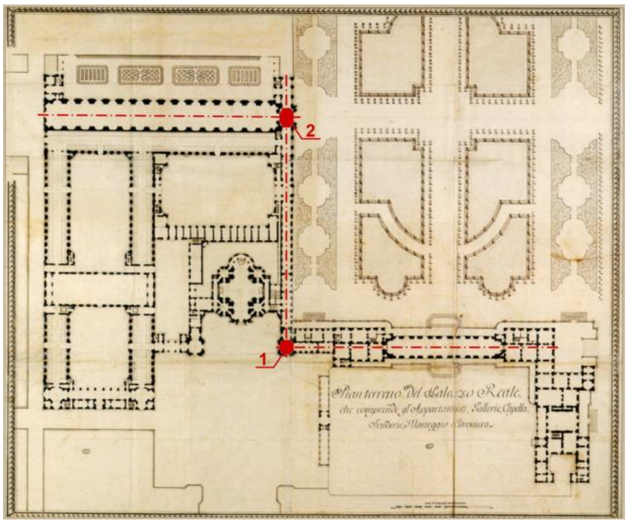

Figure 1. Graphic overlay on c. 1763. Francesco Martinez (attribute), drawing after Alfieri’s project. General plan of the complex of Venaria Reale representing the competition solution. Torino, Archivio di Stato.

Figure 2. a-b) Vault extrados; c) ground floor space; d) vault intrados.

3.1 From hidden geometries to visible geometries: extrados and accessibility

Our research was born in continuity with the research project *The King and the Origami*, a set of participative and performative guided tours, using origami models to create a direct connection between Cultural Heritage and school programs in search of inclusive educational practices between Mathematics (Geometry) and Representation through Architecture (Armand et al., 2018). That experience led us to recognize the Royal Residence of Venaria Reale as a place where the interaction with the built shape makes use of physical models to correlate built objects and their theoretical and surveyed geometries. To this extent, the extrados surface of a vault is normally inserted in the structure, supporting a floor or is hidden in crawl spaces, like rooms not designed to be accessible. Occasionally, they are visible surfaces and their rediscovery (i.e. restorations) allows to make them integral part of tours adding them value.

The exceptional accessibility to the vault under analysis, which makes its extrados visible in the spaces of horizontal distribution (Fig. 3a), makes it a unicum in Venaria. With this respect, this situation suggests the involvement of such vault among the attractions of the tour, thus we wanted to study the possibility to provide a set of explanatory tools showing the intrados/extrados relationship. Other spaces have clean and visible extrados (Fig. 3d/e), but they are not necessarily accessible to visitors, as in the case of the extrados of the Galleria Grande which has been equipped with a service walkway but in no case can be included in a tour (Fig. 3 d-e).

Figure 3. a) Venaria, Extrados of the vault, accessible; b) Casale, *S. Evasio Cathedral* archaeological path (www.vitacasalese.it); c) Venaria, extrados and construction phases under preservation, ground floor, invisible; d) Venaria, extrados of *Galleria Grande*, accessible only for maintenance activities; e) Venaria, extrados of *Rimesse*, not accessible.

Generally, by observing an extrados the attention is not diverted by decorative elements or exhibited artefacts, but it might be enchanted by the materials drawing textures, structural elements such as small vertical bricks membranes or extrados arches, elements which normally are not prefigurable by the observer from the space of visit. So, discovering a normally invisible geometry highlights the need to state a relationship between it and what is visible from the other point of view, that of intrados and this is a very complex process. In fact, the ability to create spatial relationships between spaces which are accessible at different moments of the visit, to get out of the place of visit and to observe from the outside by capturing spatially and temporally dissociated visual inputs is closely linked to one's geometric and spatial skills (Maresch and Sorby, 2021). Also, design and building geometries do not necessarily offer intrados and extrados as identical geometries with different scale factors, since they are not concentric surfaces, but they are outcome of a constructive practice which mediated between theoretical and technical knowledge and implementation. The educational approach used in the origami project, translating vaults into paper models without thickness, must be considered only a starting point to build a correlation between the intrados and the extrados (Cumino et al., 2015).

3.2 From theoretical surface to tangible surface

The process of sharing an architectural artifact involves a constructive dialogue between perceptive solicitations and critical knowledge. The architectural shape is in fact outcome of a complex process and, as such, difficult to refer to a simple shape which could be immediately recognizable and/or analytically describable (Ugo, 2008). Its description, however, can be the result of multiple approximation processes: decomposed and recomposed surfaces as sum of simple, theoretical, sometimes even symbolic ones, can more effectively support the communication of the vaulted surface to a user of the museum space if conveyed through physical and geometrically defined models. Also, these considerations suggested to draw inspiration by a long tradition in mathematical education (mainly for geometry) regarding physical objects, with historical collections of physical models of algebraic surfaces. In the second half of the nineteenth century, indeed, great progress in the study of geometry stimulated the interest of many mathematicians in the construction of such models, starting from their equations (Schilling, 1911). This activity had, as today, interactions both with research (as a means of discovering and verifying the obtained results) and with teaching at the university level, not only in Mathematics, but also in other disciplines such as Civil Engineering and Architecture (Giacardi, 2015).

3.3 Physical models to visualize and make explicit Geometry as an intangible heritage

A critical rereading of some mathematical models may be useful to explain the different design geometries and the perceptible ones, between shape, materials, and expressive potential (Fig. 4). To understand an abstract geometric shape, one would think of it as a set of points in the space satisfying certain equations, in relation with its geometric constructability, but this is not even necessary, if one has in mind the geometric definition of the entity in question. On the other hand, while solid models (Fig. 4c-d) explicitly and completely describe the surface, wire models (Fig. 4a-b) do not show it directly, they present indeed some plane sections defined by their edges only: the surfaces must therefore be perceived as passing through them. The choice of how many and which plane sections are useful for communication is crucial for the model descriptive effectiveness: for example, the model in Fig. 4a, consists of the 3 maximum sections edges, which could coincide with the 3 canonical projection planes, leading to possible misunderstandings such as the fact that 3 planes can be considered exhaustive for the surface description. Similarly, also the models described by complete plane sections (Fig. 4e-f), oblige the user to perfect the set of haptic/visual information: planes are correlated to each other until they form a 'beehive' which acts as a reference system for perceiving the surface even if it is not directly traced.

Figure 4. Tübingen Collection (Seidl et al. 2018). a-b) Wire models: H. Wiener, 1911. c-d) Solid models: c) J. Ehrhard, 1872/1876; d) R. Diesel, 1878. e-f) Card-board models by cutting planes: e) C. Tesch, 1895; f) A. von Brill, 1874. g-h) Card-board models: g) unknown Author, 1900; h) W. Schöbe, 1931.

The use of plane sections, whether defined only by edges or by complete area, offers an indirect representation of the surface,

presenting it in any case complete, without altering its geometry. In polyhedron models (Fig. 4g/h) the represented surface could also be used to approximate by defect the surface passing through all the vertices of polygons defining it, thus creating a representation that today we could easily assimilate to a mesh one.
Physical models that use section planes are suitable to represent theoretical geometries which are outcome of a survey, while string models can be used when it is already known that the theoretical surface underlying the built one is a ruled surface.

4. LIDAR SURVEY AND POINT CLOUD OF THE VAULT: CONNECTING THE INTRADOS AND THE EXTRADOS (FR, VS)

One of the goals of this study is the connection of the two surfaces of the vault, intrados and extrados, in a unique coordinate system. To allow this procedure, a control network has been surveyed by using a total station and by adopting a free network scheme.
The vertexes of the control network have been materialized by using artificial targets that can be automatically located in the point clouds acquired by laser scanning. Some of these points has been used to orient the different scans and other to check the reached final accuracy of the metric survey. For the survey of the surfaces, it was chosen to use a terrestrial laser scanner (TLS) technology, as it represents a consolidated solution in the panorama of Cultural Heritage documentation and because it offers a sufficient level of accuracy by considering the goal of the metric survey.
Multiple scans were acquired whit a Time of flight (ToF) laser scanner, the Faro Focus3d S120. This device is characterized by ± 2mm in distance measurement, a vertical and horizontal field of view of 305° and 360°. In total 20 scans were acquired of the intrados and extrados (8 scans in the intrados area and 12 scans in the extrados area), to obtain a point cloud as complete as possible. The acquisitions were planned to allow an overlap between adjacent scans greater than 30%.
After the acquisition phase, the point clouds were processed through the Faro SCENE software using a consolidated workflow (Chiabrando et al., 2016). The useful parts of the scans have been cleaned and isolated through a pre-processing step: the purpose of this step is to remove noise and unwanted points (e.g. spikes).

Block	N° of scans	IPC algorithm		Target based	
		Mean discrepancies [mm]	Discrepancies < 4 mm (%)	Mean discrepancies [mm]	Standard Deviation [mm]
INTRADOS	8	1.5	80	2.1	1.1
EXTRADOS	12	0.9	90	2.9	1.3

Table 1. Precision of the LiDAR registration process.

Target	X error [cm]	Y error [cm]	Z error [cm]
4	0.4	0.6	0.2
5	0.1	0.2	0.3
7	0.7	0.3	0.4
11	0.4	0.3	0.4
13	0.8	0.2	0.3
16	0.3	1.0	0.1
20	0.6	0.5	0.3
22	0.2	0.3	0.5
25	0.8	0.1	0.1
28	0.4	0.3	0.1
30	0.3	0.1	0.1

Table 2. Accuracy verification of the LiDAR registration process.

A first registration of intrados and extrados was carried out separately by using a procedure based on ICP (Iterative Closest Point) algorithm: this algorithm minimizes the differences between two points clouds by using a least Squares approach (e.g. all the measurements after the noise filtering could be considered free from systematic and gross errors).
The obtained point clouds of the intrados and extrados were finally connected using some vertexes of the control network (Fig. 6). The following tables show the precision and accuracy results of the LiDAR registration process.
Table 2 shows the accuracy control of the metric survey, the measures used to check the final accuracy have errors less than 1 cm. The obtained result is useful to check the mutual position of the scans.

Figure 6. LiDAR point clouds of the intrados and extrados connected together with highlighted the portions of point clouds selected for the study (V. Scolamiero)

5. DECIMATION AND OPTIMIZATION OF THE POINT CLOUD

To achieve our goal, it is necessary to interpret and analyse data deriving from the reconstructed set of point clouds in 3DReshaper environment. We identified the useful parts of the point clouds by segmenting them and maintaining only a subset containing points of the two surfaces (intrados and extrados), thus not considering points of the other parts of the Alfierian Gallery and limiting the selection to barely two meter above the springer plane of the intrados and one meter above the key point of the extrados. This operation was carried on by modelling the surfaces through a post-processing step which included the segmentation of point clouds, consisting of subdividing the 3D points into subsets. Due to the object nature, in this specific case the segmentation was done manually, and the subsets were defined by functional and structural entities of intrados and extrados (Fig. 7a-b).

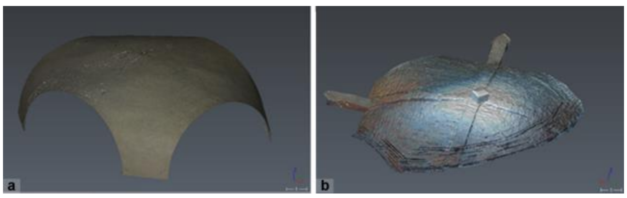

Figure 7. a) Intrados point cloud, b) extrados point cloud.

We proceeded with two separated operations: extraction of sections from the point cloud subset and its conversion into a mesh.

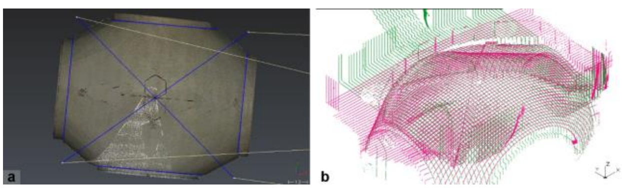

Figure 8. a) Extraction of the segmented point cloud to generate the sections. B) Vertical sections of the vault.

The two sets of vertical sections (Fig. 8) were exported into AutoCAD® and mounted together using the point 100,100,100 as a reference, to produce the orthographic projections of the object, for easily extrapolate some dimensional data.
We then created the meshes of intrados and extrados surface, by transforming the subsets by 3D modelling. Modelling method commonly used in point cloud processing software are meshing methods, in this case mesh creation is based on triangulation algorithms (Delaunay triangulation).

6. INTERPRETATION OF SURVEY DATA

As a first step, we used data from point cloud to define a set of 2D views (scale 1:100) of the entire vaulted site (Fig. 9).

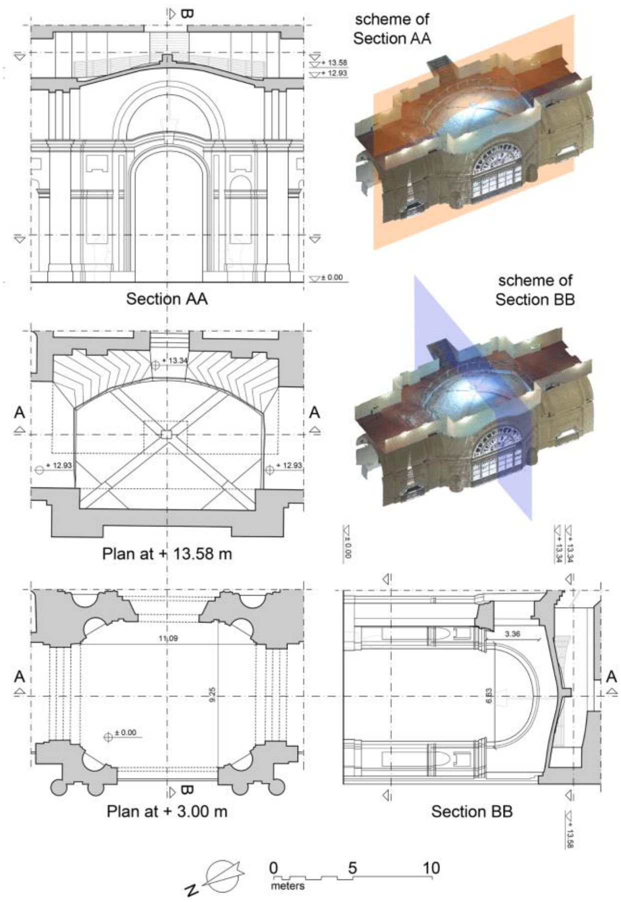

Figure 9. Graphic restitution of the hall, original scale 1:100. (drawings: M. Pavignano)

We then created two sets of vertical sections, one parallel and one perpendicular to the axis of the Alfierian Gallery, in order to analyse the vault by means of virtually cut it. This operation generated 48 longitudinal sections and 58 transversal sections.
The parameters for extraction of these sections were: thickness of sectioning planes 0.002 mm and distance between each consecutive planes 0.20 m. As shown in Fig. 8a-b, first and last sections of both sets were nearly tangent to walls, to identify the horizontal projection of the surfaces.
We used the modelled meshes to obtain two sets of horizontal sections, one for the intrados and one for the extrados.
The parameters for the extraction were: thickness of sectioning planes 0.002 mm and distance between each consecutive plane 0.10 m (Fig. 10). The first section of the intrados set coincided with the springer plane of the vault while the last one of the extrados was nearly tangent the upper portion of the structure.
Resulting sections were collected on a single AutoCAD® project and used to create the laser cut model by horizontal planes (Fig. 15a-d-g).
We decided to cut the mesh and not the point cloud, due to the scale of resulting model, set to 1:100 (Tagliari and Florio, 2013). We also sectioned the point cloud in the same way, using that sections to analyse the possible curvatures of the vault on the vertical planes (Fig. 11, 17a-c).

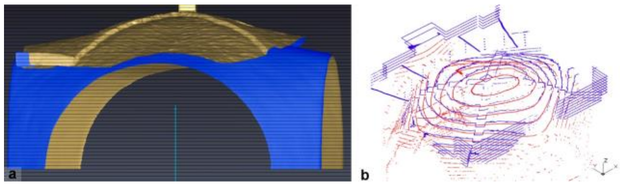

Figure 10. a) Scheme of the horizontal sections applied on the meshes; b) horizontal sections of the vault.

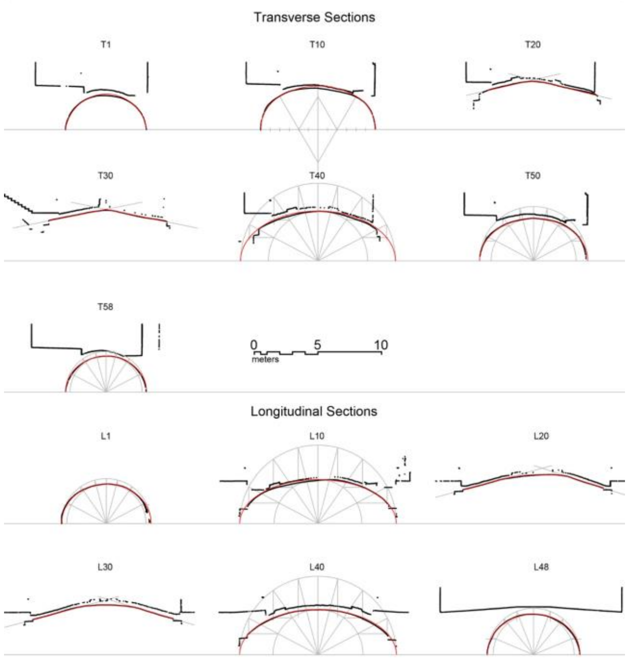

Figure 11. Examples of longitudinal and transversal sections. (drawings: M. Pavignano)

The springer plane is sited at +9.41 m over the ground level, near 0,40 m over the top level of the entablature decorating the walls. The difference between the two planes is not easily perceivable without the graphic analysis of the surveyed data. The key stone of the vault is sited at +13.58 m over the ground level. This means that the total height of the vault is around 4.17 m. Analysing the longitudinal sections, we can obtain the height of the transversal arches that delimit the vault surface on the two smaller sides, being it 12.39 and 12.48 m.

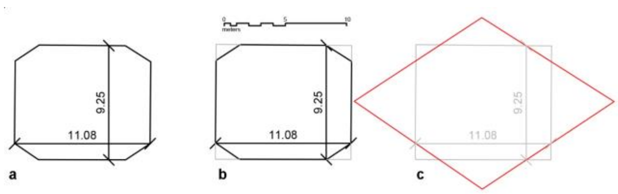

Figure 12. Reconstruction of the intrados projection. (drawings: M. Pavignano)

On the other hand, the transversal section shows the height of the longitudinal arches delimiting the vault surface on the two bigger sides at 12.67 and 12.73 m (Fig. 11).
The reconstruction of the ground floor plan shows that the vault is built on a non-regular octagon, having each couple of opposite sides parallel between them and inscribed into a rectangle and the small sides draw rhombus (Fig. 12).

7. FROM SURVEYED SURFACE TO HAPTIC MODEL

To represent not only a built surface but a vaulted structure, defined by the intrados and extrados, the approximation process to bring it back to theoretical surfaces proves to be further complex. The path from theoretical geometry to haptic model is two-way, compatibly with the level of approximation of the model production, and we could therefore theoretically start from the survey of the physical model to obtain an analytical description of the object, aware of the impact of dynamics relating to each data collection and their subsequent interpretation (Bortot, 2019).
The point of contact between the two dynamics is then represented, in our opinion, by the physical models obtained by a sequence of complete plane sections, whose intersection creates a reference grid – static or dynamic – or, in its limit case of zero distance between sections, by overlapping, in a communication method like that of level curves.
Data discretization process defines the most suitable tangible model for the description of the analysed artifact is directly correlated with the communicative potential of the model itself. A critical analysis of mathematical models has been useful in recognizing the few dynamics that can be transferred from theoretical geometrically defined surfaces modelling to the surveyed ones with all the irregularities typical of an architectural artefact.

7.1 Prototyping physical models to sharing built Geometry

We operate with models built by successive plane sections using different methods.
A laser cutter was used to cut 2mm thick vegetable cardboard sheets both to build a model with overlapping glued layers (Fig. 13a-d-g) and to create a model with plane sections hinged on comb supports that keeps them aligned and with defined distance to create the suggestion of the surface without directly describing it (Fig. 13b-e-h).
The first model, made by a stratification of vegetal cardboard layers, was derived from the sets of horizontal sections extracted from the meshes. the elevation of the model is formed by superimposing layers, thus the 2D file cannot directly deal with this issue, which is demanded to the thickness of the material used. Since we produced a 1:100 scale model made by 51 sections between them by 0,10 m, we used a 1 mm thick vegetal cardboard.
The second physical model was elaborated by using 10 vertical longitudinal sections extrapolated from the point cloud. Each section represents a vertical slice of the vault and once produced, all the elements are joined by using 4 combs to hold them in place. The model is in 1:50 scale.
A 3D printer was used to print a 1:100 scale model theoretically obtained as a continuous succession of overlapping sections until a solid model was made, lightened only to avoid wasting materials, integrated with a superstructure, immediately recognizable, useful for its realization (Fig. 13c-f-i).
The starting survey is the same, its processing for production is not, thus we followed two different approach in order to engineering them. We elaborated the file starting from the couple of meshes derived from the point cloud merging them in Rhinoceros.
Among solid models (Fig. 13d/g/f/i), the layered/glued solution, although less refined and apparently more approximate in the surface description, is the one that allows to better read relationship between the parts, emphasizing the geometries which support the shape so as to enable easy movement on it. So, the shape described by level curves makes the surface containing edges perceptible in a similar way to that described by surface models. On the contrary, the 3D laser model is much more adherent to the surface but does not offer the user a reference system to move between the intrados and extrados, forcing him to a further perceptive path, to orient himself on the surface.

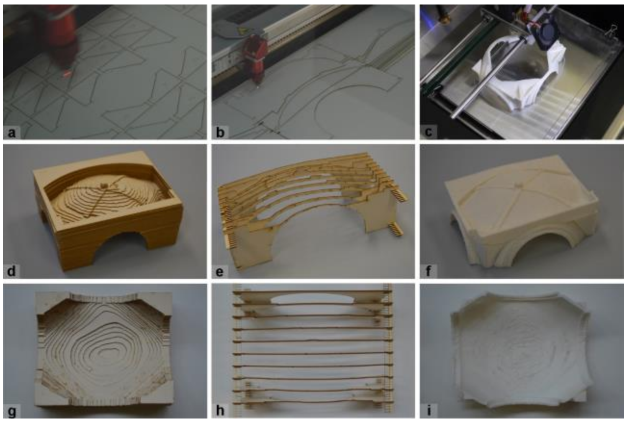

Figure 13. Haptic Models. **a/d/g)** Laser cut model by horizontal planes. **b/e/h)** Laser cut model by vertical planes. **c/f/i)** 3D printed model. **a-c)** Work procedures: **d-f)** extrados; **g-i)** intrados.

7.2 From point cloud to possible theoretical Geometry

Formulating hypothesis on the built shape of the vault we analyse our sets of point cloud sections.
By comparing profiles of sections of Fig. 11 it appears that the surface of the vault does change its sections, varying from semi-circulars to elliptical, passing from polycentric to almost linear. These sections are equal for both intrados and extrados. It is important to highlight that most of the sections do not fit perfectly to any of the proposed geometries, due to structure deformation. Observing sections in Fig. 11, suggests conjectures about the theoretical surface. At first, the shape of sections lines induces to compare the vault with cylindrical or conical surfaces. However, the use of 3DReshaper highlights the low matching between portions of the vault and these two surfaces (less than 20% by ±0.1 m). By investigating other ruled surfaces, one can think of a surface close to a conoid, as described by its geometric properties and showed by collections of mathematical models.
Here we work on ¼ of the vault (Fig. 17): first we isolate the sections T1 and L1, that pair with the horizontal plan of springer plane and the height keystone point of the vault (K) gives us the main geometrical information about the hypothetical conoidic surface. We then search for the other main elements of such surface by connection the keystone of the vault to the keystones of sections T1 and L1 (T, L) and projecting this segment to intersect the normal to the major axis of the vault (laying on m line), standing and a distance from the T1 defined by the intersection F between the prosecution of the two consequent sides (AC, BE) of the octagonal springer plan, finding G. FG is the edge of the conoidic surface while half the arch ATB is the directrix.

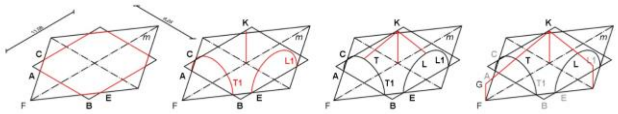

Figure 17. Geometric genesis of the possible conoidic surface. (drawings and graphic analysis: M. Pavignano)

We generate it in Rhinoceros, by setting the network of curves that lay on it with position option (Fig. 18a). We then section such surface, to obtain the ¼ of the theoretical surface, then we flip it twice on the axis of the polygonal perimeter. We check the correspondence between the modelled surface (converted to mesh) and the point cloud in 3DReshaper (Fig. 18b).

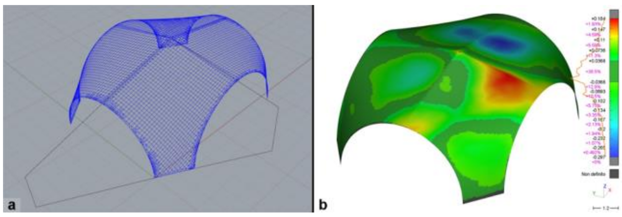

Figure 18. a) Generating the conoidic mesh; **b)** comparison between point cloud and Rhino mesh (modelling: M. Pavignano).

If we care that the built surface presents many discrepancies between possible theoretical sections and real sections, and that the main focus of this research was not to identify the best fit, we can assume that the result of this operation is quite good: we obtained that 38,5% of the intrados match the theoretical surface by ±0.0368 m, and that roughly 80% of it is between -0.102 and +0.11 m (Cipriani et al., 2013). The main issues arise where the vault presents deformity.
Looking at bibliographical references, Alfieri might have used geometries derived from Guarini’s treatise (Guarini, 1737), such as conoidic surfaces. We must point out that our case is not the same of Guarini’s (Spallone, Vitali, 2017b) it is a generalization of a conoid, as its ruling are not parallel to a fixed plane.

8. CONCLUSIONS

The exceptional accessibility condition of the extrados of our Vault, visible in a particular museum context, makes it an optimal test area for our project by enhancing visitors’ spatial visualization abilities in recognizing 3D geometry and by promoting critical shape-reading activities through haptic models. Moreover, it is a viable case study for enriching its dissemination by explicating its Geometry as qualifying intangible heritage of the same built architecture.
The dialogue between theoretical surface and relative real complex surface through an approximation process can be shared in the use of physical model, both in its meaning of object to be explored and in that of its realization design. Interacting simultaneously with physical and digital models, typical of our interdisciplinary approach, would reinforce the recognition of analogies and differences by their haptic/visual exploration.
However, it is not possible to think of translating the complexity of an architectural artefact by a geometrically and rigorously defined model: starting from the analysis of mathematical models, «naked and explicit» in Alberti’s intention (Grassi, 2007), we underline once again importance and necessity of survey procedures for reading the architectural object. Survey outcomes must be critically interpreted from an interdisciplinary approach which, in this context, leads to their materialization, translating them in models without betraying their meaning. Each model, theoretical or coming from a survey, 2D or 3D, physical or digital, has its communication features regarding users’ identities and motivations, context typologies, organizational issues (Solima, 2012).
The language was calibrated to users, with a view to a possible placing within a museum tour. The described models, those of the built shape and of the analytically defined theoretical shape nearest to the built one, are some of the prototypes born to mediate geometric language and contents emerged from the metric survey practice and its interpretation.

ACKNOWLEDGEMENTS

We warmly thank F. Diara PhD for his technical support during the survey in Venaria Reale; the Centro Studi Residenze Reali Sabaude for granting us the possibility to work inside the Royal Residence of Venaria Reale; prof. C. Cumino for her support with the geometrical analysis.
For Authors M. Pavignano and U. Zich the contribution has been developed within the research projects MAG.IA 2019-2021 and Resilience4R3C.

REFERENCES

Armand, P., Cumino, C., Pavignano, M., Spreafico M. L., Zich U., 2018. Geometry and Origami to Share Cultural Heritage: Results of the Experimentation “The King and the Origami” at the Royal Residence of Venaria. In *Bridges Stockholm 2018. Mathematics, Art, Music, Architecture, Education, Culture*. Tesselletion, Phoenix, 375-378.

Azzolino, M. C., Lacirignola, A., 2019, *Toccare il Villaggio Leumann. Comunicazione inclusiva dell'Architettura*. Torino: Politecnico di Torino

Bellini, A., 1978. *Benedetto Alfieri*. Milano: Electa.

Bortot, A., 2019. Parametric Surfaces: Mathematical, Geometric and Artistic Models. In Belardi, P. (ed.), *Proceedings of the 40th International Conference of Representation Disciplines Teachers*, Roma, Gangemi, 417-424.

Cetorelli, G., Guido, M. R., 2020. Accessibilità e fruizione ampliata al patrimonio culturale italiano. Una sfida per il Sistema Museale Nazionale. In Cetorelli, G., Guido, M. R. (eds.), *Accessibilità e patrimonio culturale. Linee guida al piano strategico-operativo, buone pratiche e indagine conoscitiva*. MIBACT, Roma, 19-26.

Chiabrando, F., Di Pietra, V., Lingua, A., Maschio, P., Noardo, F., Sammartano, G., Spanò, A., 2016. TLS Models Generation Assisted by UAV Survey. *Int. Arch. of Photogramm. Remote Sens. Spatial Inf. Sci.*, XLI-B5, 413-420. https://doi.org/10.5194/isprs-archives-XLI-B5-413-2016.

Cipriani, L., Fantini, F., Bertacchi, S., 2013. Criteri di indagine degli spazi voltati nell’ambito dell’architettura storica e in archeologia. *SCIRES-it 3 (2)*, 101-134. http://dx.doi.org/10.2423/i22394303v3n2p101

Cornaglia, P. 2006. *Giardini di Marmo Ritrovati. La Geografia del Gusto in un Secolo di Cantiere a Venaria Reale (1699-1798)*. Lindau, Torino.

Cornaglia, P., 2018. L'architettura degli Stati del re di Sardegna: trent'anni e più di cantieri, ricerche, studi e storiografia. In Raviola Blythe, A., Rosso, C., Varallo, F. (eds.), *Gli spazi sabaudi. Percorsi e prospettive della storiografia*. Carocci, Roma, 241-247.

Cumino, C., Frigerio, E., Gallina, S., Spreafico, M. L., Zich, U., 2015. Modeling vaults in origami: a bridge between mathematics and architecture. In Miura, K. et al. (eds.), *ORIGAMI 6 II. Technology, Art, Education. Proceedings of the Sixth International Meeting on Origami Science, Mathematics, and Education.* American Mathematical Society, Providence (RI), 571-582.

Cumino, C., Pavignano, M., Zich, U., 2022. *Geometrie tangibili. Catalogo visuale di modelli per la comprensione della forma architettonica | Tangible geometries. Visual catalogue of models for understanding the architectural shape*. Aracne, Roma.

Empler, T, Caldarone, A., Fusinetti, A., 2022, La stampa 3D come forma di rappresentazione per la comunicazione alla disabilità visiva. In: Càndito, C., Meloni, A. (eds.) *DAI. Il Disegno per l'Accessibilità e l'Inclusione*. Alghero: Publica press, 476-491.

Evans, R., 2000. *The Projective Cast: Architecture and Its Three Geometries*. Mit Press, Cambridge (MA).

Fatta, F., 2021. I contributi del Disegno per l'accessibilità al patrimonio architettonico. L'accessibilità nel patrimonio architettonico. In Germanà, M. L., Prescia, R. (eds.) *Approcci ed esperienze tra tecnologia e restauro | Accessibility in architectural heritage. Approaches and experiences between technology and restoration*. Conegliano (Italy): Anteferma Edizioni, 278-283.

Frommel, S. (Eds.), 2015, *Les maquettes d'architecture: fonction et évolution d'un instrument de conception et de réalisation*. Paris-Roma: Picard-Campisano Editore.

Gardner, H., 1983, *Frames of Mind: The Theory of Multiple Intelligences*. New York: Basic Books.63

Giacardi, L., 2015. Geometric Models in Mathematics Teaching in Italy at the Turn of the Twentieth Century. *Mathematisches Forschungsinstitut Oberwolfach* 47, 2784-2787.

Goldsmith, S., 1963, *Designing for Disables: A Manual of Technical Information*. London: Royal Institute of British Architects, Technical Information Service.

Grassi, G., 2007. *Leon Battista Alberti e l'architettura romana*. Franco Angeli, Milano.

Guarini, G., 1737. *Architettura civile*. Gianfrancesco Mairesse, Torino.

Hachen, M., 2007, *Scienza della Visione. Spaizo e Gestalt, design e comunicazione*. Milano: Apogeo.

Leopold, C., 2015. Visual Representations for Spatial Thinking. In: Cocchiarella, L. (ed.), *The Visual Language of Technique Vol. 3 – Heritage and Expectations in Education*. Springer, Cham, 37-56.

Maresch, G., Sorby, S. A., 2021, Perspectives on Spatial Thinking. *Journal of Geometry and Graphics*, 25(2), 271-293

March, L., 2002. Architecture and Mathematics since 1960. In Williams, K., Rodrigues J. F. (eds.), *Nexus IV: Architecture and Mathematics*. Kim Williams Books, Fucecchio, 9-33.

Ragazzo, F., 1997, Modelli di architetture: disegni e tecniche. In M. Giovannini (Eds.) *L'immagine mediata dell'architettura. Atti del seminario di studio*. Roma: Gangemi, 19-33

Rudiero, R., 2019. Comunicare la conoscenza, partecipare alla conservazione: la valorizzazione in progress di Santa Giulitta. In Demeglio, P. (ed.), *Un paesaggio medievale tra Piemonte e Liguria. Il sito di Santa Giulitta e l'alta Val Tanaro*. All'Insegna del Giglio, Sesto Fiorentino, 259-264.

Tagliari, A, Florio, W., 2013. Digital fabrication of physical models to analyse unbuilt projects using laser cutter. In Todd M.K. (ed.), *High Value Manufacturing: Advanced Research in Virtual and Rapid Prototyping*. CRC Press, Boca Raton (FL), 385-398.

Sdegno, A., Cochelli, P., Riavis, V., 2017, Modeling grimaces. Survey and haptic representation of two sculptural heads by Franz Xaver Messerschmidt. In: Di Luggo, A. et al. (Eds.) *Territories and frontiers of representation. Proceedings of the 39th International Conference of the Teacher or Representation Disciplines*. Roma: Gangemi, 969-976.

Schilling, M., 1911, *Catalog mathematischer Modelle für den höheren mathematiscen Unterricht*. Leipzig: Martin Shilling,

Seidl E., Loose, F., Bierende E. (eds.), 2018. *Mathematik mit Modellen. Alexander von Brill und die Tübinger Modellsammlun*. Museum der Universität Tübingen MUT, Tübingen.

Solima, L., 2012. *Il Museo in Ascolto. Nuove Strategie di Comunicazione per i Musei Statali*. Rubettino Editore-Ministero per i Beni e le Attività Culturali, Soveria Mannelli.

Spallone, R., Vitali, M., 2017a. Baroque Turin, Between Geometry and Architecture. *The Mathematical Intelligencer* 39(2), 76-84. https://doi.org/10.1007/s00283-017-9725-y.

Spallone, R., Vitali, M., 2017b. *Star-shaped and Planterian Vaults in Turin baroque atria*. Aracne, Ariccia.

Ugo, V. 2008. *Mìmesis. Sulla Critica della Rappresentazione dell'Architettura*. Maggioli, Sant'Arcangelo di Romagna.

Williams, K., Ostwald, M. J., 2015. The Revolutionary, The Reactionary and The Revivalist: Architecture and Mathematics After 1500. In Ostwald, M. J., Williams, K. (eds.), *Architecture and Mathematics from Antiquity to the Future. Volume II: The 1500s to the Future*. Birkhäuser, Basel, 1-27.

Williams, K., Ostwald, M. J., 2017. Manifestations of Geometry in Architecture. *Nexus Network Journal* 19(1), 1-3, DOI 10.1007/s00004-017-0332-1.

www.vitacasalese.it (last access: 15/04/2023)